Class 25, given on March 1, 2010, for Math 13, Winter 2010

We now discuss the last of the three great theorems in this class: Stokes' Theorem. Before we state the theorem, we need to explain how an oriented surface can induce an orientation on its boundary.

Recall that an orientation for a surface S is a continuous choice of unit normal vector on the entire surface, and intuitively corresponds to a choice of one of the two sides of the surface. Suppose S has a boundary which is a curve $\partial S = C$. Then the orientation S induces on C is the orientation of C which is compatible with the right-hand rule: that is, if we point our thumb in the direction of the orientation of S at points near C, then the orientation on C which is obtained is the direction in which the other four fingers point along C. Another way of saying this is that if you walk along C in the direction of the orientation induced by S with your head pointed in the direction of the unit normal for S, then S will be on your left-hand side. The textbook sometimes calls this the *positive orientation of* C induced by S.

Examples.

- If S is a region in the xy-plane, as in Green's Theorem, with upward pointing orientation, then the orientation induced by S on its boundary curve C is exactly the same with this definition as in the definition we used when discussing Green's Theorem. Indeed, we said that a simple closed curve was positively oriented if we moved in the counterclockwise direction along C, which was identical to saying that the interior of C always stayed on the left hand side of motion along C. This is compatible with the more general definition above, as one can check.
- Let S be the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$, with orientation pointing radially outward. Then the orientation induced on the boundary circle C (which is $x^2+y^2 = 1, z = 0$) is the counterclockwise orientation.

Theorem. (Stokes' Theorem) Let $\mathbf{F}$ be a C^1 vector field in $\mathbb{R}^3$ defined on some piece wise oriented surface S with boundary $\partial S = C$ which carries the orientation induced by S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S}.$$

Again, notice that this theorem has the same qualitative flavor as Green's Theorem, the Divergence Theorem, and the Fundamental Theorem of Calculus. The theorem says that, under suitable hypotheses, the surface integral of some function ($\nabla \times \mathbf{F}$) over a surface S is equal to the integral of a related function ($\mathbf{F}$) on the boundary of S.

Stokes' Theorem is not quite as easy to use as the Divergence Theorem, simply because it is harder to compute $\nabla \times \mathbf{F}$ than $\nabla \cdot \mathbf{F}$. In general, it is easier to calculate a line integral than it is to calculate the surface integral of a curl, but in certain situations (namely, when $\nabla \times \mathbf{F}$ is equal to $\mathbf{0}$), Stokes' Theorem can be used to simplify the calculation of a line integral.

Example. Let $\mathbf{F} = \langle \sin y + e^z, x \cos y, xe^z \rangle$. Evaluate the line integral of $\mathbf{F}$ across the curve C given by the ellipse $x^2 + y^2/4 = 1, z = 2$, with counterclockwise orientation.

Directly calculating this line integral would be fairly difficult, because of the somewhat complicated definition of $\mathbf{F}$. Instead, we will try using Stokes' Theorem. We start by computing $\nabla \times \mathbf{F}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ \sin y + e^z & x \cos y & xe^z \end{vmatrix} = \langle 0, 0, 0 \rangle.$$

If we choose S to be any surface with C as boundary (such as $x^2 + y^2/4 \leq 1, z = 2$), then Stokes' Theorem says

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_S 0 \, dS = 0.$$

When applying Stokes' Theorem you should be sure to check that $\mathbf{F}$ is C^1 throughout all of S, as it is in this case.

If it seems difficult to use Stokes' Theorem for calculations, this is more than made up for by the fact that Stokes' Theorem has great theoretical significance. In the above example, notice that $\mathbf{F}$ is C^1 on $\mathbb{R}^3$, and $\nabla \times \mathbf{F} = \mathbf{0}$. Recall that this means that $\mathbf{F}$ is conservative on $\mathbb{R}^3$; as a matter of fact, $f(x, y, z) = x \sin y + xe^z$ is a potential function for $\mathbf{F}$. If we knew this, we could have obtained the above result using the fact that the line integral of a conservative vector field around any closed path equals 0.

This seems to make even the above application of Stokes' Theorem obsolete, but it turns out that Stokes' Theorem is used to prove the fact that $\nabla \times \mathbf{F} = \mathbf{0}$ on $\mathbb{R}^3$ (or more generally, any simply connected region in $\mathbb{R}^3$) implies that $\mathbf{F}$ is conservative!

Examples. (Three theoretical applications of Stokes' Theorem)

- We want to use Stokes' Theorem to show that if $\nabla \times \mathbf{F} = \mathbf{0}$ for a C^1 vector field $\mathbf{F}$ on a simply-connected region D in $\mathbb{R}^3$, then $\mathbf{F}$ is conservative on D. Let C be any closed path contained in D; because D is simply connected it is possible to find a surface S which lies entirely in D whose boundary is C. Then Stokes' Theorem applied to this choice of S, C gives

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot d\mathbf{S} = \iint_S 0 \, dS = 0.$$

  That D is simply connected is needed to ensure that we can find a surface S entirely contained in D whose boundary is C. For example, if D is instead a solid torus (literally, in the shape of a donut), then one can check that D is not simply connected – for example, a circle wrapped once around the inner ring of the solid torus cannot be continually deformed to a point. If you think of various surfaces S with this circle C as boundary, you will find that every choice of S which you can think of will have to leave D somewhere, and therefore you will be unable to apply Stokes' Theorem to C since you cannot find S for which you know $\nabla \times \mathbf{F} = \mathbf{0}$ over all of S. (For proofs of these topological facts, you will want to take a course in topology.)
- Stokes' Theorem can be used to prove Green's Theorem. Recall the statement of Green's Theorem: if C is a simple closed curve in $\mathbb{R}^2$ with positive orientation, D is the interior of C, and $\mathbf{F} = \langle P, Q \rangle$ is a C^1 vector field on D, then

$$\int_C P \, dx + Q \, dy = \iint_D Q_x - P_y \, dA.$$

  To apply Stokes' Theorem to this setup, we embed this copy of $\mathbb{R}^2$ into $\mathbb{R}^3$ by declaring it to have z coordinate 0; i.e., we call this copy of $\mathbb{R}^2$ the xy plane. We can then think of S as D, with upward pointing orientation (to ensure that the induced orientation is the positive orientation on C), and $\mathbf{F} = \langle P, Q, 0\rangle$ as a vector field defined on S. In particular, $\mathbf{n} = \langle 0, 0, 1\rangle$. Thinking of $\mathbf{F}$ as now being a vector field in $\mathbb{R}^3$, we can compute $\nabla \times \mathbf{F}$:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = (Q_x - P_y)\mathbf{k}.$$

(We use the fact that P, Q are functions only of x, y, so that $P_z = Q_z = 0$.) Therefore, Stokes' Theorem applied to $S = D$ and C gives

$$\int_C P\,dx + Q\,dy = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\,dS = \iint_D Q_x - P_y\,dA.$$

Stokes' Theorem is powerful indeed if it contains Green's Theorem as a special case!

- Much like how we used the Divergence Theorem to formalize the notion of $\nabla \cdot \mathbf{F}$ as measuring the divergence of a point, we can use Stokes' Theorem to formalize the idea of curl as measuring the rotational tendency of a vector field at a point.

  If we are interested in the value of $\nabla \times \mathbf{F}$ at a point P, let S be a small circular disc of raidus r centered at P with unit normal everywhere given by a vector pointing in the same direction as $\nabla \times \mathbf{F}$. Because r is small, the value of $\nabla \times \mathbf{F}$ across S is well approximated by $\nabla \times \mathbf{F}(P)$. Then the surface integral of $\nabla \times \mathbf{F}$ across S is approximated by

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\,dS \approx \iint_S |\nabla \times \mathbf{F}(P)|\,dS = |\nabla \times \mathbf{F}(P)|\pi r^2.$$

  On the other hand, if C is the boundary of S, then Stokes' Theorem tells us the above surface integral also equals

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\,dS \approx |\nabla \times \mathbf{F}(P)|\pi r^2.$$

  Therefore, $\nabla \times \mathbf{F}(P)$ is approximately equal to

$$\nabla \times \mathbf{F}(P) \approx \frac{1}{\pi r^2} \int_C \mathbf{F} \cdot d\mathbf{r}.$$

  This approximation is accurate in the limit; that is, as $r \to 0$ the above approximation becomes an equality. The line integral on the right can be thought of as a measure of the rotational tendency of the vector field $\mathbf{F}$ in a plane orthogonal to $\nabla \times \mathbf{F}(P)$.

There is a sometimes a clever way of using Stokes' Theorem to simplify the calculation of surface integrals. First, we will use the fact (not proven in this class) that if a C^1 vector field $\mathbf{F}$ on $\mathbb{R}^3$ satisfies $\nabla \cdot \mathbf{F} = 0$, then there exists another vector field $\mathbf{G}$ such that $\nabla \times \mathbf{G} = \mathbf{F}$.

Suppose we are asked to evaluate the surface integral of such a vector field $\mathbf{F}$ across a surface S_1. It may happen that S_1 is very complicated, but that we can find another, simpler surface S_2 with identical boundary curve C. Then Stokes' Theorem tells us

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \nabla \times \mathbf{G} \cdot \mathbf{n}\,dS = \int_C \mathbf{G} \cdot d\mathbf{r} = \iint_{S_2} \nabla \times \mathbf{G} \cdot \mathbf{n}\,dS = \iint_{S_2} \mathbf{F} \cdot d\mathbf{S}.$$

That is, the value of the surface integral of $\mathbf{F}$ is independent of the choice of surface S, as long as all surfaces have the same boundary curve.

Example. Let $\mathbf{F} = \langle -2x, y, z\rangle$, and let S_1 be the hemisphere $x^2 + y^2 = 1, z \geq 0$ with radially outward pointing orientation. Evaluate the integral of $\mathbf{F}$ across S_1.

Directly calculating this integral would be annoying since we would have to use spherical coordinates to parameterize S_1. First, we check that $\nabla\cdot\mathbf{F} = -2+1+1 = 0$, and of course $\mathbf{F}$ is C^1 on $\mathbb{R}^3$. S_1 induces the counterclockwise orientation on its boundary $x^2 + y^2 = 1, z = 0$. We let S_2 be the unit disc $x^2 + y^2 \leq 1, z = 0$ with upward pointing orientation; then one immediately sees that S_2 induces the same orientation on C as S_1. Then the above discussion tells us that we can replace the evaluation of the integral across S_1 with evaluation of the integral across S_2, which is geometrically much simpler. As a matter of fact, since $\mathbf{n} = \langle 0, 0, 1\rangle$ on S_2, on S_2 we have

$$\mathbf{F} \cdot \mathbf{n} = \langle -2x, y, z\rangle \cdot \langle 0, 0, 1\rangle = z = 0.$$

Therefore, we will be integrating the 0 function on S_2, so the value of the surface integral of $\mathbf{F}$ along either S_1 or S_2 is equal to 0.

If you remember how we used the Divergence Theorem, though, you will notice that we already had a method of reducing the evaluation of the integral across S_1 to the surface S_2. Since S_1 and S_2 together bound a solid E, we can apply the Divergence Theorem to E, and since $\nabla \cdot \mathbf{E} = 0$, the Divergence Theorem also tells us that the integral across S_1, S_2 are equal to each other. Nevertheless, this shows how there seems to be a subtle relationship between Stokes' Theorem and the Divergence Theorem, despite the fact that they seem to be somewhat different from each other.